Best Hotel Problem

Largest Dowry Problem

This problem, known as the best hotel or the largest dowry, is one of the great classics of probability fun. Stated as the best hotel it goes like this.

One is driving down a highway that is known to have a hundred hotels. The driver wants to be sure to stay in the BEST one but he does not know which is best. Therefore anytime he sees a hotel he stops and checks it out. If he guesses it is not the best he goes on to the next one. However he is not allowed to go backwards. Once he has rejected a hotel he cannot return to it.

In principle the hotels could be ranked in quality 1 to 100. However the driver does not know the rank number until after he rejects it. Thus he cannot simply keep looking until he sees number 100. What is his best strategy?

Suppose one has one hundred cards numbered 1 to 100. From a shuffled deck of these one draws D cards. This group will be called the Draw. One notes the largest number L contained in the Draw. After the Draw all these cards are discarded and more cards are chosen. One keeps the first card that has a number larger than L. The object of the game is to choose D to maximize the probability that the kept card (after the draw) is number 100.

Let

$$\mathrm{A} = \mathrm{D}/100$$

Then the probability that card number 100 is inside the Draw equals A. In this case the game is lost. He has passed the best one. Let

$$\mathrm{B} = 1 - \mathrm{A}$$

is given by

$$BxA.$$

In this case the game is won since 100 will be the first card above 99 that one can encounter.

Now the probability that both 100 and 99 are not in the draw but 98 is is

$$B^2xA.$$

Now one wins half the time but only half the time since two cards bigger than 98 are available after the draw but in two possible orders , 100,99 or 99,100. In the first case one wins but in the second case one looses because 99 is encountered before 100. Thus the combined probability of winning in this situation is

$$B^2xA/2$$

Continuing one finds the probability that 100 ,99, and 98 are not in the draw but 97 is is

$$B^3xA.$$

Here one wins only a third of the time since any of three cards bigger than 97 follow the draw 100 must come first to win. Thus the combined probability of winning in this situation is

$$B^3xA/3$$

Continuing this way and adding the winning probabilities one finds the total

$$Ax(B + B^2/2 + B^3/3 + B^4/4 + B^5/5......)$$

By a simple Maclaurin expansion, the series in the parenthesis is equal to $-\ln(1 - B)$ where ln is the natural logarithm.

But

$$1 - B = A$$

so that

So that

$$\text{Prob} = Ax(-\ln A)$$

Note that $-\ln A$ is a positive number since A is less than one.

Now to maximize the Probability we take the derivative of the above recalling that the derivative of $\ln A$ is $1/A$

$$d\text{Prob}/dA = A\ (-1/A) - \ln A = -1 - \ln A$$

Setting this equal to zero one has

$$\ln A = -1$$

$$A = e^{-1}$$

Thus the best Draw is $D = 100/e$.

This is easily checked by monti carlo simulation.