WHAT IS THIS THING CALLED e ?

One of the most frequently asked but poorly answered questions in the mathematics of physics or chemistry or physiology is, " What is this thing called e^{R*t} or e^{-R*t} ".

It occurs, for example, as the growth of the number of bacteria on a culture plate where the rate of growth is proportional to the number present at any time.

$$\frac{d\ N}{d\ t} = R * N \tag{1}$$

Or in the second form as the decay of radioactivity in a sample of a single radioactive isotope.

$$\frac{d\ Q}{d\ t} = -R * Q \tag{2}$$

1

The usual classroom answer is that e is the base of the natural logarithms which is no answer at all.

The more informative but lengthy answer is given in the typical advanced calculus course requires complicated explanations of series convergence etc.

Hopefully the following provides a clear answer requiring at most the first two weeks of an elementary calculus course.

In each of the above examples one needs to find a quantity that has the property that its rate of change is proportional to it's value.

Consider the following series.

$$\text{Series} = \text{S} = 1 + \alpha * \text{t} + (\alpha * \text{t})^2/2! + (\alpha * \text{t})^3/3! + (\alpha * \text{t})^4/4!.... \tag{4}$$

Calculate $\frac{\text{d Series}}{\text{dt}}$ one term at a time.

The first term's derivative with respect to t is

$$\frac{\text{d}(1)}{\text{dt}} = 0 \tag{5}$$

the next term's derivative

$$\frac{\text{d}(\alpha * \text{t})}{\text{dt}} = \alpha \tag{6}$$

is the first term of the series S multiplied by α.

Then since $\frac{\text{d}(\text{t}^2)}{\text{dt}} = 2 * \text{t}$

$$\frac{(\text{d}(\alpha * \text{t})^2/2!}{\text{dt}} = \alpha^2 * \text{t} \tag{7}$$

the second term of the series S multiplied by α.

The next term is

$$\frac{(\text{d}(\alpha * \text{t})^3/3!}{\text{dt}} = \alpha^3 * \text{t}^2/2! \tag{8}$$

is the third term of the series S multiplied by α.

Thus

$$\frac{d\ S}{d\ t} = \alpha S \tag{10}$$

S now meets the requirement of being proportional to it's own derivative.

There is a special symbol for the given series. It is called $e^{\alpha * t}$ where e is a special number to be raised to the $\alpha * t$ power. The reason it is represented as a number raised to a power is that this allows the rules of elementary algebra to be used to find the product of two such series.

For example let

$$S_a = 1 + a * t + (a * t)^2/2! + (a * t)^3/3!... \tag{11}$$

and

$$S_b = 1 + b * t + (b * t)^2/2! + (b * t)^3/3!... \tag{12}$$

These would individually satisfy Eq. 1

$$\frac{d\ S_a}{d\ t} = aS \qquad \text{and} \qquad \frac{d\ S_b}{d\ t} = bS \tag{13}$$

We can define a third series S_p which is the product of the two above and show that it will also satisfy the requirment of being proportional to its own derivative. One could do this by brute force by multilplying series S_a by series S_b term by term. However there is a much easier way.

Define the product $S_p = S_a \ * S_b$

Then using the usual rule for the derivative of a product [2]

$$d\ (S_p) \qquad d\ (S_a * S_b) \qquad d\ S_b \qquad d\ S_a$$

$$S_a * b\ S_b + S_b * a\ S_a = (a+b)S_a * S_b = (a+b) * S_p \tag{15}$$

Thus the derivative of S_p which is the product of S_a and S_b is proportional to the product multiplied by a constant which is the sum of a and b. If we let S_a be represented by a number raised to a power a and S_b be represented by the same number raised to the power b then the product will be that number raised to the power a + b. However, the special number e is not arbitrary because it has to agree with the series when $\alpha * t = 1$.

To find the value of the number e let $\alpha\ * t = 1$ and simply add terms of the series Eq.4 Unfortunately when one tries to sum the series, one finds the sum depends on how many terms are included.

Number of terms.		Sum
2	$1+1$	$= 2$
3	$1+1+\frac{1}{2!}$	$= 2.5$
5	$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}$	$= 2.708333333...$
10	$1+1+\frac{1}{2!}+........+\frac{1}{9!}$	$= 2.71828152...$
20	$1+1+\frac{1}{2!}+........+\frac{1}{19!}$	$= 2.7182818285...$

Notice that as the number of terms increases the calculated sum keeps getting larger. Since all the terms are positive this will continue. The more terms included, the bigger the sum.[3]

However the size of the steps keeps getting smaller so that between ten terms and twenty terms the sum changes only in the seventh place after the decimal point. Since the series S has an infinite number of terms there is no guarantee that summing a finite number of terms will ever yield a good enough value.

greater than two, provides a sum which is a lower bound such that e is known to be greater than that number.

Now one needs some way to find an upper limit or upper bound of the possible value of S. Then S would be trapped between the lower bounds calculated above and an upper limit to be found below.

The following series T resembles S but can be easily summed and, as shown below, is always larger than S.

Call this series T.

$$1 + \beta + \beta^2 + \beta^3 + \beta^4 + \beta^5 ... = 1/(1 - \beta) = \mathrm{T} \qquad (16)$$

[4] This series, although infinite (as was S) has a definite sum given on the right provided that β is positive and less than one. [5]

Compare series S to series T with $\alpha * \mathrm{t} = \beta$.

$$\mathrm{S} = 1 + \beta + \beta^2/2! + \beta^3/3! + \beta^4/4! ...... = \mathrm{e}^{\beta} \qquad (17)$$

$$\mathrm{T} = 1 + \beta + \beta^2 + \beta^3 + \beta^4 + \beta^5 ...... = 1/(1 - \beta) \qquad (18)$$

Each term (other than the first) of T is a power of β. Each term (other than the first) of S has the same power of β as the corresponding term of T but is divided by a factor which is always greater than or equal to one. Therefore S is always less than or equal to T as long as β is greater than zero. Thus one has

$$\mathrm{e}^{\beta} < 1/(1 - \beta) \qquad (19)$$

But not quite out of the woods yet. One cannot get the value of e directly because that would require β equal to one for which the term $1/(1 - \beta)$ becomes infinite

However one can get the upper limit of the square root of e by letting $\beta = \frac{1}{2}$.

$$e^{1/2} < 1/(1 - \frac{1}{2}) = 2 \quad (20)$$

which can be squared yielding

$$(e^{1/2})^2 = e < 2^2 = 4 \quad (21)$$

This is now an upper limit for e but not a very good one. It is too big. But suppose one sets $\beta = 0.1$ now one has

$$e^{0.1} < 1/(1 - 0.1) = 1.11111.... \quad (22)$$

Raise both sides to the tenth power.

$$e = (e^{0.1})^{10} < (1/(1 - 0.1))^{10} = (1.11111....)^{10} = 2.867... \quad (23)$$

A much better value for a limit. e is trapped between the 20 term approximation for S which was 2.7182818285..... and 2.867.....above. Now try $\beta = 0.01$

$$e = (e^{0.01})^{100} < (1/(1 - 0.01))^{100} = (1.010101...)^{100} = 2.731..... \quad (24)$$

or $\beta = 0.001$

$$e = (e^{0.001})^{1000} < (1/(1-0.001))^{1000} = (1.001001001...)^{1000} = 2.7196422....($$

Now e is known to be between 2.7182818285 and 2.7196422... By carrying both of these procedures to more places one can converge on the classical value for the mysterious e=2.718281828 only to Eq. 1 where the right side of the equation has a positive sign leading to a series of all positive terms in powers of β. In Eq. 2 the right side is negative which leads to negative values for β so that odd power terms in the series are negative. We can get around this complication by relating the Q of Eq. 2 to the N of Eq. 1 as follows.

Let Q be defined by $NQ = 1$.

Calculate the derivative of the product

$$\frac{d(NQ)}{dt} = N * \frac{d\ Q}{dt} + Q * \frac{d\ N}{dt} = 0 \tag{26}$$

Now let $N = N_0 e^{R*t}$ which is the solution to Eq. 1 multiplied on both sides by a constant N_0.

Thus

$$\frac{d\ N}{dt} = R * N_0 e^{R*t} \tag{27}$$

Replacing N by $N_0 e^{R*t}$ and $\frac{d\ N}{dt}$ by $R * N_0 e^{R*t}$ in Eq.26 one has

$$N_0 e^{R*t} * \frac{d\ Q}{dt} + Q * R * N_0 e^{R*t} = 0 \tag{28}$$

So that canceling terms one has

$$\frac{d\ Q}{dt} = -Q * R \tag{29}$$

which is Eq. 2

One might properly ask if there is any thing else that has the property that it's rate of change is proportional to itself.

If R and N are both real numbers (as opposed to imaginary or complex numbers) nothing other than

$$e^{R*t} \tag{31}$$

or multiples thereof has this property.

However if R is allowed to be a imaginary number new possibilities arise.

Let

$$R = i\theta \tag{32}$$

where θ is real.

Now we have

$$\frac{d\ Z}{d\ t} = i\theta\ Z \tag{33}$$

Reasoning by analogy, replace the R in Eq. 30 by $i\theta$ which yields

$$Z = Z_0\ e^{i\ \theta\ t} \tag{34}$$

where Z_0 is an arbitrary constant.

If one simply follows the rules of differentiation This obviously works. However one is still faced with the question of what one means by raising something to an imaginary power. Clearly it is not a simple extension of the primitive idea of multiplying the number by itself n times to get the n th power such as $7^3 = 7 * 7 * 7$. It is a different and strange idea.

There is a clue. (Although it is going to take a while to exploit it.) Suppose one repeats the operation of raising a real number to an imaginary power. By the usual rules of algebra

One has the extraordinary result that a real number 7 raised to an imaginary power 7^{i} and then raised to an imaginary power again $(7^{i})^{i}$ yields a real (not an imaginary) result. In particular the final result is the reciprocal of the original number.

Let us apply similar logic to Eq. 33 by taking the derivative of the derivative.

$$\frac{d}{d\ t}\left(\frac{d\ Z}{d\ t}\right) = \frac{d}{d\ t}(i\ \theta\ Z) = (i\theta)^2 Z = -\theta^2\ Z \tag{36}$$

Recall from elementary calculus that

$$\frac{d\ \sin(\theta\ t)}{d\ t} = \theta * \cos(\theta\ t) \qquad \text{and} \qquad \frac{d\ \cos(\theta\ t)}{d\ t} = -\theta * \sin(\theta\ t) \tag{37}$$

[6] and therefore

$$\frac{d^2\ \sin(\theta\ t)}{d\ t^2} = -\theta^2 * \sin(\theta\ t) \tag{38}$$

$$\frac{d^2\ \cos(\theta\ t)}{d\ t^2} = -\theta^2 * \cos(\theta\ t) \tag{39}$$

Either the sine or the cosine satisfies Eq. 36 but neither by itself satisfies Eq. 33 (repeated below for reference)

$$\frac{d\ Z}{d\ t} = i\theta\ Z$$

However by combining the sine and cosine solutions one can construct a quantity Z_{prime} that will satisfy both Eqs. 33 and 36 as follows.

which one can test by calculating

$$\frac{\mathrm{d}\ \mathrm{Z}_{\mathrm{prime}}}{\mathrm{d}\ \mathrm{t}} = -\theta\ \sin(\theta\ \mathrm{t}) + \mathrm{i}\ \theta\ \cos(\theta\ \mathrm{t}) = \mathrm{i}\ \theta\ \mathrm{Z}_{\mathrm{prime}} \tag{41}$$

.

Now find the derivative of Eq. 34 reproduced below.

$$\mathrm{Z} = \mathrm{Z}_0\ \mathrm{e}^{\mathrm{i}\ \theta\ \mathrm{t}}$$

which was originally constructed so that it would obey the formal rules of differentiation. Applying this to Eq. 34 yields

$$\frac{\mathrm{d}\ \mathrm{Z}}{\mathrm{d}\ \mathrm{t}} = \mathrm{Z}_0\ \mathrm{i}\ \theta\ \mathrm{e}^{\mathrm{i}\ \theta\ \mathrm{t}} = \mathrm{i}\ \theta\ \mathrm{Z} \tag{42}$$

which is the same result obtained in Eq. 41.

Z and $\mathrm{Z}_{\mathrm{prime}}$ clearly satisfy the same differential equation. Therefore except for the possible addition of a constant they must be equal. By comparing the values of Z and $\mathrm{Z}_{\mathrm{prime}}$ at $\mathrm{t} = 0$, we find the constant must be zero from which we conclude that

$$\mathrm{e}^{\mathrm{i}\ \theta\ \mathrm{t}} = \cos(\theta\ \mathrm{t}) + \mathrm{i}\ \sin(\theta\ \mathrm{t}) \tag{43}$$

This is the clue to raising any real number to an imaginary power.

Let $\theta\ \mathrm{t} = \mathrm{n}$ be any real number.

$$\mathrm{n} = \mathrm{e}^{\ln\ \mathrm{n}} \tag{44}$$

7

The last step is from Eq. 43.

Example: Let n = 4.81 Then ln n is 1.57 or approximately $\pi/2$. Thus

$$1.57^{\mathrm{i}} = \mathrm{i} \tag{46}$$

one has a real number raised to an imaginary power yielding an imaginary result.

Suppose one has a complex number Z to be raised to the i th power the result of which we will call $\mathrm{Z^i}$. Any complex number can be represented in the form

$$\mathrm{Z} = \mathrm{R\ e^{i\phi}} \tag{47}$$

where R is a real number.

$$\mathrm{Z^i} = (\mathrm{R\ e^{i\phi}})^{\mathrm{i}} = \mathrm{R^i}\ (\mathrm{e^{i\phi}})^{\mathrm{i}}) = \mathrm{e}^{-\phi}\ \mathrm{R^i} \tag{48}$$

Since R is real, $\mathrm{R^i}$ can be found from Eq. 45 as

$$\mathrm{R^i} = (\mathrm{e^{\ln\ R}})^{\mathrm{i}} = (\mathrm{e})^{\mathrm{i\ln\ R}} = \tag{49}$$

$$\cos(\ln\ \mathrm{R}) + \mathrm{i}\ \sin(\ln\ \mathrm{R})$$

Replacing $\mathrm{R^i}$ in Eq. 48 by Eq. 49 above and $(\mathrm{e^{i\phi}})^{\mathrm{i}}$ by $\mathrm{e}^{-\phi}$ yields

$$\mathrm{Z^i} = (\mathrm{R\ e^{i\phi}})^{\mathrm{i}} = \mathrm{e}^{-\phi}(\cos(\ln\ \mathrm{R}) + \mathrm{i}\ \sin(\ln\ \mathrm{R})) = \tag{50}$$

$$\mathrm{Z^i} = \mathrm{e}^{-\phi}\mathrm{e}^{\mathrm{i}\ \arctan\ \frac{\sin(\ln\ \mathrm{R})}{\cos(\ln\ \mathrm{R})}} = \mathrm{e}^{-\phi}\mathrm{e}^{\mathrm{i\ln\ R}}$$

In the same way that Z is written in Eq. 47 as a radius and an angle, Z^i in Eq. 50 above defines a new radius

$$r = e^{-\phi} \tag{51}$$

and a new angle

$$\beta = \arctan\left(\frac{\sin(\ln R)}{\cos(\ln R)}\right) = \ln R \tag{52}$$

Thus

$$Z^i = r\, e^{i\,\beta} = e^{-\phi}\, e^{i\,\ln\,R} \tag{53}$$

To find $(Z^i)^i$:

$$(Z^i)^i = (e^{-\phi}\, e^{i\,\ln\,R})^i = (e^{-i\,\phi}\, e^{-\ln\,R}) = \tag{54}$$

$$\frac{1}{R}\,(e^{-i\,\phi})$$

Recall from Eq. 47 that

So that the product

$$Z\ (Z^{i})^{i} = Z\ Z^{-1} = 1 \quad \text{or} \quad (Z^{i})^{i} = \frac{1}{Z} \tag{55}$$

as we hoped it would. Thus the rule for calculating the i th power of a complex number, $Z = R\ e^{i\phi}$ is given by Eq. 50

$$Z^{i} = e^{-\phi} e^{i\ \arctan\ \frac{\sin(\ln\ R)}{\cos(\ln\ R)}} = e^{-\phi} e^{i \ln\ R} \tag{56}$$

Calculating Z to a complex power will be left as an exercise for the reader.

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